MATHEMATICAL MISCELLANEA

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19. Solutions of the Quadratic Equation.

Our *Miscellanea 5. A Geometric Solution of the Quadratic Equation*[1] brought us several interesting pieces of mail.

Professor Carl B. Boyer of Brooklyn College wrote to call our attention to Euclid's geometrical solutions of the quadratic. These solutions are to be found in Book II, Propositions 5, 6, 11 and Book VI, 27, 28, 29 of Euclid's *Elements*. The best known of these is II, 11 which solves $x^2+ax-a^2=0$ (a regarded as positive). This is the algebraic form of the division of a line in "extreme and mean ratio" and hence is the "golden section" which yields the "golden ratio" named the "divine proportion" by Pacioli (1509) and utilized in recent discussions of "dynamic symmetry." This problem is discussed in many places. Perhaps the most recent is an article by Carnahan[2] which is titled *Geometric Solutions of Quadratic Equations.*

Problems 85, 86 of Euclid's *Data* are also geometric equivalents of the solution of quadratics.

At this point, it should be noted that John Wallis' third solution which we gave in *Miscellanea 5*, figure 5, page 280 is, essentially identical with that given by René Descartes in *La Géométrie. Livre Premier* to be found on page 302 of his *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences* (Leyden: 1637). Also, it would take only a slight modification of Wallis' first method (our figure 3, page 279, *loc. cit.*) to make it identical with the solution given by Descartes on page 303 of his work.

A letter from Willian J. Hazard of the Department of Engineering Mathematics of the University of Colorado includes the following list of 18 ways to solve $ax^2+bx+c=0$ taken from an article which he published in January 1924 in the *Colorado Engineer:*

1. By factoring by inspection.
2. By factoring after a substitution, $z=ax$, which leads to $z^2+bz+ac=0$.
3. By factoring in pairs by splitting bx into two terms.
4. By completing the square when a is 1 and b is even.
5. By completing the square as usual after dividing through by a.
6. By completing the square by the Hindu method ("the pulverizer"), i.e. by multiplying through by $4a$ and adding b^2 to both sides.
7. By completing square as given, adding $b^2/4a$.
8. By the formula.
9. By trigonometric methods (see Wentworth-Smith, *Plane Trigonometry*).
10. By slide rule (see Joseph Lipka, *Graphical and Mechanical Computation.* John Wiley and Sons, Inc. [1918], p. 11 ff.)
11. By graphing for real roots. (All modern textbooks.)
12. By graph, extended for complex roots. (See: Howard F. Fehr, "Graphical Representation of Complex Roots," *Multi-Sensory Aids in the Teaching of Mathematics, Eighteenth Yearbook of the National Council of Teachers of Mathematics* [1945] pp. 130–138. George A. Yanosik, "Graphical Solutions for Complex Roots of Quadratics, Cubics, and Quartics," *National Mathematics Magazine*, 17 [Jan. 1943], pp. 147–150.)

[1] The Mathematics Teacher, vol. 43 (Oct. 1950), pp. 279–280.

[2] Walter H. Carnahan, "Geometric Solutions of Quadratic Equations," *School Science and Mathematics*, 47 (Nov. 1947), pp. 687–692.

13. Real roots by Lill circle. (d'Ocagne, *Calcul graphique et nomographie*, from which L. E. Dickson got his reference to it in his *Elementary Theory of Equations.*) (Also see J. W. A. Young's *Monographs on, Topics in Modern Mathematics* "Constructions with Ruler and Compasses.")
14. By extension of the Lill circle to include complex roots.
15. Using the graph of $y=x^2$ and $y=-bx-c$ to find real roots. (Lipka, *op. cit.* p. 26, modifies and extends this solution; Schultze, *Graphic Algebra;* Hamilton and Kettle, *Graphs and Imaginaries.*)
16. By extending (15) to include complex roots (Hamilton and Kettle, Schultze).
17. By use of a table of quarter squares. This is a practical method of handling an equation having large constants, as we already have the table in print (Jones' *Mathematical Tables*).
18. By use of "Form Factors."

Professor Hazard adds that methods 12, 14, 17, 18 are original with himself, and that 13, 14 and 17 will be discussed in his book *Algebra Notes* to be published soon. He also suggests a 19th method:

"Make a template of transparent plastic, very slightly smaller and parallel to the unit parabola $y=x^2$, so that the pencil or ruling pen will draw the true curve. Scratch co-ordinates on it for accurate placing, and for any equation move the template to a position parallel to the Y axis and with its vertex at the point

$$\left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right).$$

While this can't be called a different 'method' of solution, I think it should be included as a very handy and practical 'variation of approach' to the graphic solution of the quadratic in one unknown."

We hope that Professor Hazard and others will extend our list and include expositions of some of the less obvious or less easily available procedures.

Before leaving this topic, however, we should mention the reprint sent us by Professor H. D. Larsen of "Solutions of the Quadratic Equation" which appeared in the Spring 1950 issue of *The Pentagon.* Professor Larsen, editor of the journal, writes that he has some copies of the article which he will sell for thirty-five cents each. The article was written by Raymond H. Gillespie as an Honors Thesis in Mathematics while he was an undergraduate at Albion College, Albion, Michigan where Professor Larsen is also Chairman of the Mathematics Department. This thesis includes an historical survey, early solutions (including Viete's solution via the substitution of $x=u+z$ in $x^2+ax+b=0$ and the equating of the coefficient of u to 0), derivations of the formula (including that via determinants by Euler and Bezout), methods of factoring, graphical solutions, determination of imaginary roots, and solutions by trigonometric methods. Although the tabulation of procedures is not, of course, absolutely complete and the historical survey ignores entirely recent discoveries in regard to Babylonian achievements, nevertheless it is well done and contains much interesting material which would enrich both classroom teaching and club programs.

20. Notes on Werner Todd's Trisection

Professor Norman Anning handed us some notes on *Miscellanea 4. Trisecting Any Angle* (THE MATHEMATICS TEACHER, vol. 43 (Oct. 1950), pp. 278–279), with the admonition to destroy his message and then to keep away from trisections as we promised earlier. We can no more destroy them without some mention than he could resist the temptation to give them to us.

The curve which Mr. Todd constructed is one half of the Trisecant of Delanges[3]

[3] Described by P. Delanges in *La trisegnante nuova curva* (Verona: 1783) according to Gino Loria writing in *Enciclopedia delle Mathematiche*

a quartic curve of which a cartesian equation is $(y^2-4m^2)(x^2+y^2)+4m^2=0$.

Professor Anning further notes that this method reappeared in 1943 in the work of a William Joseph Frederick of Battle Creek, Michigan, who constructed the curve by a clever combination of compasses and a T-square. The year before this the same curve was used for this purpose by Harvey P. Sleeper (Harvard, 1942, Eliot House). Sleeper's mechanism is to be found on the end-paper of an early printing of R. C. Yates, *The Trisection Problem* (1942) an excellent reference on this topic which is still available from the author at the United States Military Academy, West Point, New York.

Elementari e Complementi, volume II, part II, (Milano: 1938), p. 394 ff. See also Heinrich Wieleitner, *Spezielle Ebene Kurven* (Leipzig: 1908), pp. 79–82.

21. Inequalities, the Quadratic, and the "Ambiguous Case."

The "ambiguous case" in the solution of oblique triangles is, of course, the case in which two sides and an angle opposite one of them are given. The various situations in which 0, 1, or 2 solutions occur are usually analyzed from two drawings, one of the acute- and the other of the obtuse-angled case. This is desirable visualization.

Analytically, in the acute angled case where sides a and b and angle A are given, the altitute $h_c=b \sin A$ is computed and the nature of the solutions determined by comparing h_c with a. This is convenient since $b \sin A$ is simultaneously the natural first step in determining B via the Sine Law. If A is obtuse, $a>b$ is necessary and sufficient to guarantee a unique solution.

Another approach which may be used as an exercise or extra credit project for the pedagogical purpose of making an interesting display of the interconnections between this problem, the quadratic discriminant, and conditional inequalities is as follows.

As above consider a, b, and A to be given. Then $a^2=b^2+c^2-2bc \cos A$ may be considered as a quadratic equation in c. From this

$$c=b \cos A \pm \sqrt{b^2 \cos^2 A-b^2+a^2}.$$

Since negative or zero values for a, b, c have no geometric significance, we see that (1) if $\cos A$ is positive (i.e. if A is acute) there may be 0, 1, or 2 solutions while (2) if $\cos A$ is negative (i.e. if A is obtuse) there will be one and only one solution if and only if $b^2 \cos^2 A-b^2+a^2 > b^2 \cos^2 A$ which condition simplifies to merely the requirement that $a>b$.

In situation (1) there will be no solution if $b^2 \cos^2 A-b^2+a^2<0$. But $b^2 \cos^2 A-b^2 = -(b \sin A)^2$. Hence the original condition reduces to the requirement that $b \sin A>a$ or that $h_c>a$. From this it readily follows that there will be a single unique solution if $b \sin A=h_c=a$, and if $h_c<a$ there will be two solutions as long as $b \cos A>\sqrt{b^2 \cos^2 A-b^2+a^2}$. There will be only a single solution however if $h_c<a$ and $b \cos A \leqq \sqrt{b^2 \cos^2 A-b^2+a^2}$. This latter condition is easily seen to be equivalent to $b \leqq a$.

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There will be study groups for elementary, junior high and senior high school teachers on arithmetic and high school mathematics. In addition there will be study groups on enrichment materials and the slow learner. Leaders for these groups are outstanding specialists from various parts of the country. Also there are to be general sessions. Some of the best talent in the field of the teaching of mathematics and other fields of interest will speak at these sessions. A new feature will be a laboratory where models will be demonstrated and made. Rooms and meals will be available on the campus at reasonable rates. Everything possible is being done to insure that everyone will have a grand time together and a profitable experience. Plan now to attend this second L.S.U. Mathematics Institute. A printed program will be available after March 1st. For further information, write Houston T. Karnes, Department of Mathematics, Louisiana State University, Baton Rouge 3, Louisiana.